Lattice Codes and Generalized Minimum Distance Decoding for OFDM Systems

Alan Clark and Desmond P. Taylor
Department of Electrical and Computer Engineering
University of Canterbury
Christchurch, New Zealand
Email: a.clark@elec.canterbury.ac.nz

March 9, 2005

Abstract

Lattice coding of orthogonal frequency division multiplexing (OFDM) systems is considered. Mapping of multilevel construction lattices to OFDM blocks is shown, and a methodology for probabilistic analysis of multistage generalized minimum distance (GMD) decoding of the received OFDM blocks is derived. As a case study transmission of points from a 128-dimensional Barnes-Wall lattice is considered. Tight approximations to the system error rate are obtained and verified by simulation. It appears that GMD decoding of lattice encoded OFDM provides high coding gain at low complexity.

Contents

1 Introduction

Orthogonal Frequency Division Multiplexing (OFDM) systems [1,2] are well suited to lattice codes. Given an $\frac{N}{2}$ subcarrier OFDM system transmitting two dimensional subcarrier signal points, the OFDM block may be elegantly represented as a single point, $\mathbf{x}$, in N dimensional Euclidean space. Lattice coding of the OFDM block simply requires restriction of $\mathbf{x}$ such that it is an element of some lattice Λ, or an equivalent sphere packing. Since OFDM systems typically employ a large number of subcarriers (48 or more) we may use high dimensional lattices, with large coding gain. However, decoding of high dimensional lattices using maximum likelihood decoding is not feasible. We consider lattices defined by multilevel constructions, which are elegantly decoded using multistage decoding. Multistage Generalized Minimum Distance (GMD) decoding is an attractive low complexity approach [3]. We show that encoding within OFDM blocks can provide large coding gain, at low complexity, while avoiding the latency of coding over succesive blocks.

Although analysis of GMD decoding for Binary Phase Shift Keying (BPSK) modulated codes in additive white Gaussian noise (AWGN) exists [4], we usually transmit Quadrature Amplitude Modulation (QAM) [5] within an OFDM block sent in a frequency selective channel. Therefore, we address the problem of mapping lattice points to sets of $\frac{N}{2}$ QAM constellations. We then extend the probabilistic analysis of [4] to QAM transmission. We analyse the performance of GMD decoded lattice codes on frequency selective channels, and obtain a tight approximation to the error rate for an arbitrary number of subcarriers, constellation size, channel response and SNR.

The following section briefly reviews lattice construction, OFDM signalling and the mapping of lattice points to OFDM blocks. Section 3 reviews lattice decoding and specifically GMD lattice decoding. In Section 4 we extend existing probabilistic analysis of GMD decoding to OFDM transmission of lattices mapped to QAM constellations. We derive tight bounds on the error rate of transmission over AWGN channels and frequency selective channels, for arbitrary system parameters. System simulations verifying the analysis are shown in Section 5 while the final section summarises the paper.

2 Lattice Coding and OFDM Signalling

We first provide a brief summary of lattice coding and construction. We also review OFDM transmission across AWGN and frequency selective channels. Readers unfamiliar with lattice coding are referred to [3, 6, 7], while OFDM transmission is examined by [1, 2, 8].

An N-dimensional lattice, Λ, is a discrete additive subgroup of the Euclidean space $\mathbb{R}^N$. Given an (N, K, D) binary linear block code, $\mathbb{C}$, we may define a lattice $\Lambda_{\mathbb{C}}$ using construction A of [7] as

$$\Lambda_{\mathbb{C}} = \left\{ \mathbf{x} \in \mathbb{Z}^N : \mathbf{x} \equiv \mathbf{c}(\text{mod-}2), \mathbf{c} \in \mathbb{C} \right\} \ , \tag{1}$$

where $\mathbb{Z}^N$ is the integer lattice of all integer N-tuples. We can apply construction A in m levels, to obtain construction C of [7, 9] [1]

$$\begin{aligned}\Lambda &= \bigcup_{\mathbf{c_1} \in \mathbb{C}_1 \ldots \mathbf{c_m} \in \mathbb{C}_m} 2^m \mathbb{Z}^N + 2^{m-1}\mathbf{c_m} + 2^{m-2}\mathbf{c_{m-1}} + \ldots + 2\mathbf{c_2} + \mathbf{c_1} \ , \\ &= \bigcup_{\substack{\mathbf{c_1} \in \mathbb{C}_1 \ldots \mathbf{c_m} \in \mathbb{C}_m \\ \mathbf{c_{m+1}}, \mathbf{c_{m+2}}, \ldots \in \mathbb{C}_{m+1}}} 2^m(\mathbf{c_{m+1}} + \mathbf{c_{m+2}} + \ldots) + 2^{m-1}\mathbf{c_m} + 2^{m-2}\mathbf{c_{m-1}} + \ldots + 2\mathbf{c_2} + \mathbf{c_1} \ ,\end{aligned} \tag{2}$$

where $\mathbb{C}_1, \ldots, \mathbb{C}_m$ are binary linear block codes with length N, and $\mathbb{C}_{m+1}$ is the trivial $(N, N, 1)$ binary linear block code with identity generator matrix. Although we may construct lattices from other partitions, as in *generalized construction A* of [3], throughout this paper we concern ourselves only with construction C lattices. Construction C lattices necessarily form subgroups of the integer lattice $\mathbb{Z}^N$.

2 .1 OFDM Signalling

An OFDM signal is the superposition of a number of mutually orthogonal subcarriers, spaced ΔfHz apart. We assume the number of subcarriers is $\frac{N}{2}$, for some positive even integer N. The k^{th} subcarrier signal is described by the function

$$g_k(t) = \begin{cases} \exp(j2\pi k \Delta f t) & \text{for } 0 < t < T_s + T_g \\ 0 & \text{otherwise} \ ; \end{cases} \tag{3}$$

where $T_s = \frac{N}{\Delta f}$ and T_g is a guard interval which, if chosen to be longer than the maximal channel delay spread, limits interblock interference, such that we consider each subchannel to have a flat

[1] Note that [3, 4] name this *multilevel construction A*.

frequency response. The OFDM block duration is then $T = T_s + T_g$. We modulate each subcarrier with a two dimensional symbol $S_{n,k}$, where $k = 0, 1, \ldots, \frac{N}{2} - 1$ is the subcarrier index and n denotes the time interval index. We refer to the superposition of the $\frac{N}{2}$ modulated subcarriers in the n^{th} time interval as the n^{th} OFDM block, written as

$$s_n(t) = \frac{1}{\sqrt{\frac{N}{2}}} \sum_{k=0}^{\frac{N}{2}-1} S_{n,k} g_k(t - nT) \qquad \text{for } nT \leq t < (n+1)T \ . \tag{4}$$

Sampling this signal at rate $\frac{N}{2}\Delta f$ yields discrete samples at times t_i, spaced $\frac{2}{N\Delta f}$ apart,

$$s_{n,t_i} = \frac{1}{\sqrt{\frac{N}{2}}} \sum_{k=0}^{\frac{N}{2}-1} S_{n,k} \exp\left(j2\pi \frac{t_i k}{(\frac{N}{2})}\right) \ , \tag{5}$$

which is simply the inverse discrete Fourier transform (IDFT) of the $\frac{N}{2}$ subcarrier symbols $S_{n,0}$, $\ldots, S_{n,\frac{N}{2}-1}$. Low pass filtering of these samples yields the continuous OFDM signal to be transmitted.

At the receiver we perform a discrete Fourier transform (DFT) of the rate $\frac{N}{2}\Delta f$ samples of the received signal, $r_{n,t_i}, t_i = 0, \ldots, \frac{N}{2} - 1$, to obtain noisy, attenuated modulation symbols

$$R'_{n,k} = \frac{1}{\sqrt{\frac{N}{2}}} \sum_{t_i=0}^{\frac{N}{2}-1} r_{n,t_i} \exp\left(-j2\pi \frac{t_i k}{(\frac{N}{2})}\right) \ . \tag{6}$$

In the case of a slowly fading, dispersive channel the channel transfer function is approximately constant over the symbol duration, such that $R'_{n,k} = H_{n,k}.S_{n,k} + W'_{n,k}$. Where $H_{n,k} \in \mathbb{R}^2$ is the effective channel gain and $W'_{n,k} \in \mathbb{R}^2$ is a zero-mean, Gaussian random variable with variance $\frac{N_0}{2}$ in each dimension. We assume time and frequency synchronisation, and sufficient guard interval T_g as noted above. For the AWGN channel we assume that $H_{n,k} = 1, \forall n, \forall k$. Otherwise, if we assume perfect channel state information (CSI) at the receiver, that is knowledge of $H_{n,k}, \forall n, \forall k$, then we may equalize each received symbol to obtain

$$R_{n,k} = \frac{R'_{n,k}}{H_{n,k}} = S_{n,k} + W_{n,k} \ , \tag{7}$$

where $W_{n,k}$ is a zero mean Gaussian random variable with variance $\frac{N_0}{2|H_{n,k}|}$ in each dimension. For each time interval n, we thus obtain $\frac{N}{2}$ AWGN corrupted complex modulation symbols at the receiver, comprising an OFDM received subcarrier symbol block.

2 .2 Lattice Coded OFDM Transmission

Lattices have infinite cardinality, and therefore, we must choose some finite cardinality subset, $\Lambda_f \subset \Lambda$ as a signal constellation. The choice of Λ_f affects peak to average power ratio, transmission complexity and receiver complexity, among other considerations. The work of [10,11] documents the effects of constellation choice for multidimensional signalling, and is applicable to OFDM lattice transmission. Our choice is restricted since we assume that each two dimensional modulation symbol, $S_{n,k}$ is from a single M^2-ary QAM constellation, for some positive even integer M. We denote this two dimensional QAM constellation as C_2. This restriction increases compatibility with existing standards and systems [5,12], while ensuring that practically useable two dimensional constellations are considered. We require a mapping $q : \Lambda_f \to C_2^{\frac{N}{2}}$, from the N-dimensional lattice subset Λ_f to some subset C_N of the $\frac{N}{2}$-fold Cartesian product of the subcarrier QAM constellations, $C_2^{\frac{N}{2}}$. While C_N is not necessarily a lattice, it is a finite cardinality sphere packing, which exhibits the same centre density, kissing number and coding gain as Λ_f, if and only if $q(\mathbf{x})$ may be written as a simple scaling, rotation and translation of Λ_f. That is

$$q(\mathbf{x}) = a\mathbf{R}\mathbf{x} + \mathbf{T} \ , \forall \mathbf{x} \in \Lambda_f \ , \tag{8}$$

such that a is a scalar, $\mathbf{R}$ is some fixed orthogonal $N \times N$ matrix, and $\mathbf{T}$ is some fixed $1 \times N$ vector containing a single scalar, $\mathbf{T} = [b, b, \ldots, b]$.

Assuming we are mapping to M^2-ary QAM constellations, we set Λ_f to be the subset of Λ defined by

$$\Lambda_f = \{\mathbf{x} = (x_1, x_2, \ldots, x_i, \ldots, x_N) : 0 \leq x_i < M; \mathbf{x} \in \Lambda\} \ . \tag{9}$$

Therefore we are using a cubic constellation, [10], and all points are contained within an N-dimensional cube of side length M. Although this affords no shaping gain and does not reduce the peak to average power ratio, the restriction to square QAM subcarrier constellations limits us to this choice. The required mapping from Λ_f to $\frac{N}{2}$ M^2-ary QAM constellations is essentially equivalent to a mapping to N M-ary PAM constellations, denoted C_1^N, since we may consider each QAM constellation as the Cartesian product of two PAM constellations, that is $C_2 = C_1 \times C_1$. We use the mapping $q : \Lambda_f \to C_1^N$,

$$q(\mathbf{x}) = 2\sqrt{E_0}\mathbf{x} - \left[-(M-1)\sqrt{E_0}, \ldots, (M-1)\sqrt{E_0}\right] \ , \tag{10}$$

Hence each lattice dimension is mapped to an M-ary PAM constellation,

$$C_1 = \left\{-(M-1)\sqrt{E_0}, -(M-3)\sqrt{E_0}, \ldots, -\sqrt{E_0}, +\sqrt{E_0}, \ldots, (M-3)\sqrt{E_0}, (M-1)\sqrt{E_0}\right\}, \tag{11}$$

and each OFDM subcarrier transmits points from an M^2-ary QAM constellation, C_2, with minimum energy $2E_0$ and average energy $E_{av} = \frac{2(M^2-1)E_0}{3}$ [8].

As an example, we consider a 64 subcarrier OFDM system transmitting 256-QAM points from a lattice code based on the 128 dimensional Barnes-Wall lattice, denoted BW_{128} [7,13]. In an abuse of notation we construct a sphere packing using Reed-Muller codes, [9], with construction C and refer to this as BW_{128}. Although this construction does not strictly speaking yield a Barnes-Wall lattice it does result in an equivalent sphere packing with the same coding gain [3, 7]. That is

$$BW_{128} = \bigcup_{\mathbf{c_1}\in\mathbb{C}_1,\mathbf{c_2}\in\mathbb{C}_2,\mathbf{c_3}\in\mathbb{C}_3} \left\{8\mathbb{Z}^{128} + 4\mathbf{c_3} + 2\mathbf{c_2} + \mathbf{c_1}\right\} \; ; \tag{12}$$

such that $\mathbb{C}_1$, $\mathbb{C}_2$ and $\mathbb{C}_3$ are the $(128, 8, 64)$, $(128, 64, 16)$, $(128, 120, 4)$ Reed-Muller codes respectively. Since $M = 16$, we restrict the signal constellation to the N-dimensional cube with opposite vertices at $\{0, \ldots, 0\}$ and $\{15, \ldots, 15\}$. Our finite lattice subset may then be expressed as

$$\Lambda_f = \bigcup_{\mathbf{c_1}\in\mathbb{C}_1,\ldots,\mathbf{c_4}\in\mathbb{C}_4} \{8\mathbf{c_4} + 4\mathbf{c_3} + 2\mathbf{c_2} + \mathbf{c_1}\} \; ; \tag{13}$$

where $\mathbb{C}_4$ is the $(128, 128, 1)$ code, and $8\mathbf{c_4} \in 8\mathbb{Z}^{128}$. In order to choose a point $\mathbf{x} \in \Lambda_f$ we input blocks of 128,120,64 and 8 data bits to encoders for $\mathbb{C}_4$,$\mathbb{C}_3$,$\mathbb{C}_2$ and $\mathbb{C}_1$ respectively, obtaining four 128-bit codewords. This lattice code therefore has rate $\frac{128+120+64+8}{4\times128} = 0.625$. Points are then mapped to a 16-PAM constellation with the mapping $q(\mathbf{x}) = 2\sqrt{E_0}\mathbf{x} - [15\sqrt{E_0}, \ldots, 15\sqrt{E_0}]$. Pairs of 16-PAM points are combined in quadrature to form 64 256-QAM symbols for modulation onto 64 OFDM subcarriers.

3 Lattice Decoding

Construction C lattices are easily decoded using multistage decoding, [14]. Given a block of equalized OFDM signals, $R_{n,k} = S_{n,k} + W_{n,k}, k = 0, \ldots, (\frac{N}{2} - 1)$, representing some noise corrupted lattice point, we first apply the inverse mapping to (10), $q^{-1} : C_2^{\frac{N}{2}} \rightarrow \Lambda_f$. We then obtain an AWGN corrupted, construction C lattice point,

$$\mathbf{r} = \mathbf{x} + \mathbf{w} = \bigcup_{\mathbf{c_l} \in \mathbb{C}_l, l=1,\ldots,m} \left\{ 2^m \mathbb{Z}^N + 2^{(m-1)} \mathbf{c_m} + \ldots + 2\mathbf{c_2} + \mathbf{c_1} \right\} + \mathbf{w} \ , \tag{14}$$

where $\mathbf{x} \in \Lambda_f$ and $\mathbf{w} = \{w_1, \ldots, w_N\}$ is a vector of independent zero mean Gaussian random variables. We now consider decoding a single block, and simplify notation by omitting the the block (time) index, k. We estimate $\mathbf{x}$ by finding successive estimates for $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_m}$, and at the last stage estimating the uncoded bits from the point in $2^m \mathbb{Z}^N$. More specifically, given $\mathbf{r}$ we first find the closest point, $\hat{\mathbf{c}}_\mathbf{1}$, in the coset $\bigcup_{\mathbf{c_1} \in \mathbb{C}_1} \{2^m \mathbb{Z}^N + \mathbf{c_1}\}$. Next we find the closest point, $\hat{\mathbf{c}}_\mathbf{2}$ to $\mathbf{r}$ in the coset $\bigcup_{\mathbf{c_1} \in \mathbb{C}_1} \{2^m \mathbb{Z}^N + \hat{\mathbf{c}}_\mathbf{1} + 2\hat{\mathbf{c}}_\mathbf{2}\}$. Generally, for the l^{th} stage we find the codeword $\hat{\mathbf{c}}_\mathbf{l} \in \mathbb{C}_l$, such that $\mathbf{r}$ is closest to the coset $\bigcup_{\mathbf{c_1} \in \mathbb{C}_1} \{2^m \mathbb{Z}^N + \hat{\mathbf{c}}_\mathbf{1} + 2\hat{\mathbf{c}}_\mathbf{2} + 2^{l-1}\hat{\mathbf{c}}_\mathbf{l}\}$. At the final stage we find the point $\mathbf{x_{m+1}} \in 2^m \mathbb{Z}^N + \hat{\mathbf{c}}_\mathbf{1} + 2\hat{\mathbf{c}}_\mathbf{2} + \ldots + 2^{m-1}\hat{\mathbf{c}}_\mathbf{m}$ closest to $\mathbf{r}$.

We have stated that we find the closest points to the received point, implying the use of maximum likelihood (ML) decoding. However, maximum likelihood decoding requires the consideration of every possible point, before selection of the closest point. As lattice dimension, and the length of the component codes, increases, the ML approach requires exponentially increasing complexity. Therefore, we use GMD decoding at each stage. GMD decoding of lattices codes is the subject of [3] and [4], and was shown to be an excellent low complexity approach for decoding high dimensional lattice codes, with near ML performance.

3 .1 Generalized Minimum Distance Decoding

Although GMD decoding can be applied to any group code, [3], [15], we limit discussion to binary linear block codes. Given a codeword $\mathbf{c} = \{c_1, \ldots, c_N\}$ from an (N, K, D) code $\mathbb{C}$, and some noise corrupted version $\mathbf{r} = \{r_1, \ldots, r_N\}$ of this codeword, GMD estimates $\mathbf{c}$ using hard decisions $\mathbf{u} = \{u_1, \ldots, u_N\}$ of each respective codeword element, and a vector of reliabilites $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_N\}$, corresponding to each hard decision.

Given an (N, K, D) code, $\mathbb{C}$, $\mathbf{u}$ and $\boldsymbol{\alpha}$ we perform a series of algebraic errors and erasures decoding trials, with the s least reliable symbols of $\mathbf{u}$ erased; for each $s \in \mathcal{K}$. Here $\mathcal{K}$ is a *trial enumerator set*, defined as $\mathcal{K} = \{1, 3, \ldots, D-1\}$ for even D and $\mathcal{K} = \{0, 2, \ldots, D-1\}$ for odd D. Therefore, $\lfloor \frac{D+1}{2} \rfloor$ errors and erasures trials are performed. For a code with minimum distance D, a codeword is produced if and only if $\mathbf{u}$ and $\mathbf{c}$ differ in t unerased positions, such that $2t+s < D$. Each trial thus produces a candidate codeword, $\tilde{\mathbf{c}}$, or a decoding error, such that at most $\lfloor \frac{D+1}{2} \rfloor$ candidate codewords are produced. We refer to this as the *algebraic decoding step.* The GMD decoder then chooses one of the candidate codewords as its output, in the *Euclidean space selection step.* Following [3], the decoder chooses the candidate codeword with the smallest squared generalized distance[2] from $\mathbf{u}$, defined for the AWGN channel as

$$\delta^2(\tilde{\mathbf{c}}) = \delta^2(\tilde{c}_1) + \delta^2(\tilde{c}_2) + \ldots + \delta^2(\tilde{c}_N)$$
$$\text{with } \delta(\tilde{c}_i) = \begin{cases} 1 - \alpha_i & \text{if } \tilde{c}_i = u_i \\ 1 + \alpha_i & \text{if } \tilde{c}_i \neq u_i \ . \end{cases} \tag{15}$$

The AWGN reliability metric, α_i, is defined as follows: for each received coordinate, r_i, the receiver front end finds the closest and second closest possible lattice coset coordinates, y_i and y_i' respectively. From [3], the hard decision output is $u_i = y_i$, with the corresponding reliability calculated as the projection of $(r_i - y_i)$ onto $(y_i' - y_i)$ subtracted from one. That is

$$\alpha_i = \begin{cases} 1 - \frac{\langle (r_i - y_i), (y_i' - y_i) \rangle}{|y_i - y_i'|} & \text{for } \langle (r_i - y_i), (y_i' - y_i) \rangle < |y_i - y_i'| \\ 1 & \text{for } r_i - y_i < 0 \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product. Therefore, if $r_i = y_i$ then reliability $\alpha_i = 1$, whereas if r_i lies on the decision boundary between y_i and y_i', then reliability is lowest, at $\alpha_i = 0$. Note that $0 \leq \alpha_i \leq 1$. Analysis of GMD using this AWGN metric [3, 4], shows that the generalized distance, (15) , is a lower bound to Euclidean distance, and that GMD decoding achieves bounded distance decoding.

We apply GMD decoding at each stage, $l = 1, \ldots, m$ of the lattice decoding, requiring calculation of a hard decision $\mathbf{u}_l$ and corresponding reliability $\boldsymbol{\alpha}_l$ for each stage, where $\mathbf{u}_l$ will be an element of

[2] Referred to as a *generalized distance* since this is not a true distance [15]

the appropriate lattice coset at each stage. We thus obtain a codeword estimate, $\hat{\mathbf{c}}_{\mathbf{l}}$ for each coded stage. Uncoded stages may be estimated with algebraic decoding, which is equivalent to GMD decoding of the $(N, N, 1)$, 'constituent codes' associated with these stages, since GMD decoding of a unity minimum distance block code entails a single decoding trial with no erasures.

4 Frequency Selective Channel Performance

We now extend the analysis of [4] to M-ary PAM (M^2-ary QAM) transmission over a frequency selective channel. We obtain an approximation to the error rate of multistage GMD decoding lattice encoded OFDM points, that appears to be an upper bound for the cases of interest. We transmit a point $\mathbf{x}$ in some finite subset Λ_f of an N-dimensional lattice, mapped to $\frac{N}{2}$ OFDM M^2-ary QAM subcarriers. After equalization and inverse mapping we obtain a noise corrupted lattice subset point, $\mathbf{r} = \mathbf{x} + \mathbf{w}$, where $\mathbf{x} \in \Lambda_f$; as in (14). For the AWGN channel, $\mathbf{w} = \{w_1, \ldots, w_N\}$ is a vector of iid zero mean Gaussian random variables. However, for the frequency selective channel the $\{w_i\}$ are independent, zero mean Gaussian random variables with variance dependent on the subchannel over which the lattice coordinate x_i was transmitted. Following the inverse mapping to (10) and the equalization of (7) we write the variance of w_i as

$$\sigma_i^2 = \frac{N_0}{2} . \frac{1}{(2\sqrt{E_0})^2} . \frac{1}{|H_{\lfloor \frac{i}{2} \rfloor}|^2} \text{ , for } i = 1, \ldots, N \ . \tag{17}$$

Unlike the AWGN channel, each codeword position is perturbed by noise of different variance, and the probability of correct estimation of each codeword position therefore varies.

4 .1 Single Stage Performance

Using the methodology of [4], we analyse the performance of GMD decoding of an (N, K, D) binary linear block code transmitted over some frequency selective channel. The receiver front end produces a hard decision vector $\mathbf{u} = (u_1, u_2, \ldots, u_N)$, and vector of corresponding reliabilities, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$. We denote $\mathcal{E}_b$ as the event that there are b hard decision errors. Furthermore, we let $\mathcal{I}_f = (f_1, f_2, \ldots, f_b) \subseteq \{1, 2, \ldots, N\}$ be the set of codeword positions corresponding to incorrect hard decisions, and $\mathcal{I}_g = (g_1, g_2, \ldots, g_{N-b})$ be the complimentary set of $(N-b)$ indices corresponding to correct hard decisions. We write the probability of hard decision error on the i^{th} subcarrier as p_i, that is the probability that $u_i \neq c_i$. For the i^{th} codeword position $p_i = \Pr(u_i \neq c_i) = \frac{M}{M-1} \operatorname{erfc}(\sigma_i)$, where $\operatorname{erfc}(() .)$ is the Gaussian error function, and σ is given in (17). Since the probability of error differs for each hard decision, we write the probability of exactly b hard decision errors as

$$\Pr(\mathcal{E}_b) = \sum_{\forall \mathcal{I}_f, \mathcal{I}_g} p_{f_1} p_{f_2} \ldots p_{f_N} \, . \, [1 - p_{g_1}] [1 - p_{g_2}] \ldots \left[1 - p_{g_{N-b}}\right] \tag{18}$$

where the summation is over all $\binom{N}{b}$ sets $\mathcal{I}_f$ and $\mathcal{I}_g$. The probability mass function of (18) is easily seen to be a Poisson binomial distribution [16] with parameters $\{N, p_1, p_2, \ldots, p_N\}$. For large N the number of terms in the summation of (18) becomes very large. However, the Poisson binomial distribution is accurately approximated, with known total variation, by either the binomial [17] or Poisson distributions [16], depending on the distribution of $p_1, \ldots, p_N$ [18]. Thus, the SNR range of the channel of interest will determine the better approximation. Although the binomial distribution is often a better approximation when the variance of the error probabilities $p_1, p_2, \ldots, p_N$ is close to $\frac{1}{N}\sum_{i=1}^{N}\{p_i\}\left(1 - \frac{1}{N}\sum_{i=1}^{N}\{p_i\}\right)$, here we assume channels with large dynamic range and consequently apply the Poisson approximation [18] for large N, so that

$$\Pr\left(\mathcal{E}_b\right) \approx \frac{\lambda^b e^{-b}}{b!} \text{ , with } \lambda = \sum_{i=1}^{N} p_i \ . \tag{19}$$

For the special case of the AWGN channel, or equivalently a channel with no SNR variation, the probability of error, p_i, is the same for all hard decisions, and (18) simplifies to

$$\Pr\left(\mathcal{E}_b\right) = \binom{N}{b} p_i^b \left[1 - p_i\right]^{N-b} \ . \tag{20}$$

We denote the reliability statistics of the erroneous hard decisions as $\{\alpha_{f_1}, \alpha_{f_2}, \ldots, \alpha_{f_b}\} \subseteq \boldsymbol{\alpha}$. We rank these in nondecreasing order to obtain $\beta_1, \ldots, \beta_b$, such that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_b$. Similarly, we denote the ordered reliability statistics corresponding to correct hard decisions as $\gamma_1, \ldots, \gamma_{N-b}$, such that $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{N-b}$.

We denote the event that the algebraic decoding stage produces the correct codeword when b hard decision errors occur, as $\mathcal{S}_b$. The event of a successful GMD *algebraic decoding step* is denoted $\mathcal{S}$, such that $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_N$, with the complimentary event of *algebraic decoding step* failure denoted $\mathcal{F}$. If the number of errors, $b \leq t = \lfloor \frac{D-1}{2} \rfloor$, then correct decoding is certain, while if the number of errors $b \geq D-1$ then correct decoding is impossible. Therefore we may write

$$\begin{aligned}
\Pr\left(\mathcal{F}\right) &= 1 - \Pr\left(\mathcal{S}\right) \\
&= 1 - \left[\Pr\left(\mathcal{S}_0\right) + \Pr\left(\mathcal{S}_1\right) + \ldots + \Pr\left(\mathcal{S}_N\right)\right] \\
&= 1 - \left[\Pr\left(\mathcal{E}_0\right) + \ldots + \Pr\left(\mathcal{E}_t\right) + \Pr\left(\mathcal{S}_{t+1}\right) + \ldots + \Pr\left(\mathcal{S}_{D-1}\right)\right] \ .
\end{aligned} \tag{21}$$

Following [4], we now calculate lower bounds upon the $\Pr\left(\mathcal{S}_b\right)$ for $b = t+1, t+2, \ldots, D-1$, and thus upper bound $\Pr\left(\mathcal{F}\right)$.

The *algebraic decoding step* requires a number of errors and erasures decoding trials, with k erasures made for all $k \in \mathcal{K}$, the aforementioned *trial enumerator set*. Letting the event $\mathcal{S}_{b,k} \subset \mathcal{S}_b$ denote production of the correct codeword when k erasures are made and b errors are present, we may write $\mathcal{S}_b = \bigcup_{\forall k \in \mathcal{K}} \{\mathcal{S}_{b,k}\}$. It can then be shown [4], that a tight lower bound is given by,

$$\Pr(\mathcal{S}_b) \geq \max_{k \in \mathcal{K}} \{\Pr(\mathcal{S}_{b,k})\} \ , \tag{22}$$

since the events $\mathcal{S}_{b,1}, \mathcal{S}_{b,2}, \ldots$, are highly correlated.

We let $\mathcal{U}_{\tau,k}$ denote the event that τ or more hard decision errors are erased when k erasures are made, requiring $\tau \leq k$ and $b \geq \tau$. Note that if τ or more errors are erased, this requires $\beta_\tau < \gamma_{k-\tau+1}$, that is the τ^{th} smallest reliability associated with a hard decision error must be less than the $(k-\tau+1)^{\text{th}}$ smallest reliability associated with a correct hard decision, so that at most only $(k-\tau)$ correct hard decisions are erased. The probability of $\mathcal{U}_{\tau,k}$ occurring, given that there are b hard decision errors is therefore the probability that $\beta_\tau > \gamma_{k-\tau+1}$. This is readily calculated given the pdf of β_τ, denoted $f_{\beta_\tau}(x)$ and the pdf and cdf of $\gamma_{(k-\tau+1)}$, denoted $f_{\gamma_{(k-tau+1)}}(x)$ and $F_{\gamma_{(k-tau+1)}}(x)$ respectively. These probability functions are discussed in the next subsection. We may then write

$$\Pr(\mathcal{U}_{\tau,k}|\mathcal{E}_b) = \int_0^\infty f_{\beta_s}(x) \int_x^\infty f_{\gamma_{(k-s+1)}}(y)\, dy\, dx = \int_0^\infty f_{\beta_s}(x) \left[1 - F_{\gamma_{(k-s+1)}}(x)\right] dx \ . \tag{23}$$

Presuming τ errors are erased, leaving $b-\tau$ unerased errors, a correct codeword is produced if and only if $k + 2(b-\tau) < d$, or equivalently $\tau > b - \frac{k-d}{2} \geq b - \lfloor \frac{d-k-1}{2} \rfloor$, that is, the event $\mathcal{U}_{\tau,k}$. We may therefore write

$$\begin{aligned} \Pr(\mathcal{S}_{b,k}) &= \Pr(\mathcal{U}_{\tau,k} \cap \mathcal{E}_b) \ , \\ &= \Pr(\mathcal{U}_{\tau,k}|\mathcal{E}_b) \Pr(\mathcal{E}_b) \ , \\ &= \Pr(\mathcal{E}_b) \int_0^\infty f_{\beta_\tau}(x) \left[1 - F_{\gamma_{(k-s+1)}}(x)\right] dx \ , \end{aligned} \tag{24}$$

with $\tau = b - \lfloor \frac{d-k-1}{2} \rfloor$. We can therefore upper bound the probability of GMD *algebraic decoding step* failure (21), by calculating $\Pr(\mathcal{S}_{b,k})$ for all k to lower bound $\Pr(\mathcal{S}_b)$ for all $b = t+1, \ldots, D-1$. The order statistic distributions required to calculate (24) are described in the next subsection. Note that, since we use approximations, (19) and later (28), we cannot label our analysis an upper bound on the probability of GMD error. Strictly speaking we have derived an approximation on the probability of GMD decoding error, which appears, from the results in Section 5 , to be an upper bound in many cases of interest.

4 .2 Reliability Order Statistics

The analysis thus far has generally followed that of [4]. For a frequency selective channel the pdf's of the reliability order statistics are significantly more difficult to evaluate. Recall, the reliability statistics associated with incorrect hard decisions are denoted $\alpha_{f_1}, \alpha_{f_2}, \ldots \alpha_{f_b}$, while those associated with correct hard decisions are denoted $\alpha_{g_1}, \alpha_{g_2}, \ldots, \alpha_{g_{(N-b)}}$. The order statistics associated with incorrect hard decisions are denoted, in nondecreasing order, $\beta_1, \beta_2, \ldots, \beta_b$, while those associated with correct hard decisions are similarly denoted $\gamma_1, \gamma_2, \ldots, \gamma_{N-b}$.

Since the post equalization noise variance, σ_i^2 differs across the received symbols, each α_{f_i} is independent but non-identically distributed, with pdf and cdf $f_{\alpha_{f_i}}(x)$ and $F_{\alpha_{f_i}}(x)$ respectively, as calculated in Appendices C and E for the case of M-ary PAM transmission. Given the indices of incorrect hard decisions, $\mathcal{I}_f$, we then use a result of [19] to write the pdf of the s^{th} smallest α_{f_i} given that b hard decision errors are made, that is the pdf of β_s, as

$$f_{\beta_s}(x|\mathcal{I}_f) = \frac{1}{(s-1)!(b-s)!} \operatorname{Per} \begin{array}{|cccc|l} F_{\alpha_{f_1}}(x) & F_{\alpha_{f_2}}(x) & \ldots & F_{\alpha_{f_b}}(x) & \multirow{3}{*}{$\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\}(s-1)\text{ rows}$} \\ \vdots & \vdots & \ddots & \vdots & \\ F_{\alpha_{f_1}}(x) & F_{\alpha_{f_2}}(x) & \ldots & F_{\alpha_{f_b}}(x) & \\ f_{\alpha_{f_1}}(x) & f_{\alpha_{f_2}}(x) & \ldots & f_{\alpha_{f_b}}(x) & \\ 1-F_{\alpha_{f_1}}(x) & 1-F_{\alpha_{f_2}}(x) & \ldots & 1-F_{\alpha_{f_b}}(x) & \multirow{3}{*}{$\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\}(b-s)\text{ rows}$} \\ \vdots & \vdots & \ddots & \vdots & \\ 1-F_{\alpha_{f_1}}(x) & 1-F_{\alpha_{f_2}}(x) & \ldots & 1-F_{\alpha_{f_b}}(x) & \end{array} \tag{25}$$

where $\operatorname{Per}|A|$ is the permanent [20] of a square matrix, A. Similarly, the pdf of the s^{th} smallest reliability associated with a correct hard decision, $f_{\gamma_s}(x|\mathcal{I}_g)$, is equal to

$$\frac{1}{(s-1)!(N-b-s)!} \operatorname{Per} \begin{array}{|cccc|l} F_{\alpha_{g_1}}(x) & F_{\alpha_{g_2}}(x) & \ldots & F_{\alpha_{g_{(N-b)}}}(x) & \multirow{3}{*}{$\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\}(s-1)\text{ rows}$} \\ \vdots & \vdots & \ddots & \vdots & \\ F_{\alpha_{g_1}}(x) & F_{\alpha_{g_2}}(x) & \ldots & F_{\alpha_{g_{(N-b)}}}(x) & \\ f_{\alpha_{g_1}}(x) & f_{\alpha_{g_2}}(x) & \ldots & f_{\alpha_{g_{(N-b)}}}(x) & \\ 1-F_{\alpha_{g_1}}(x) & 1-F_{\alpha_{g_2}}(x) & \ldots & 1-F_{\alpha_{g_{(N-b)}}}(x) & \multirow{3}{*}{$\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\}(N-b-s)\text{ rows}$} \\ \vdots & \vdots & \ddots & \vdots & \\ 1-F_{\alpha_{g_1}}(x) & 1-F_{\alpha_{g_2}}(x) & \ldots & 1-F_{\alpha_{g_{(N-b)}}}(x) & \end{array} \tag{26}$$

where $f_{\alpha_{g_i}}(x)$ and $F_{\alpha_{g_i}}(x)$ are the pdf and cdf of the reliability associated with a correct hard decision, given the correct hard decision indices, $\mathcal{I}_f$. The functions $f_{\alpha_{g_i}}(x)$ and $F_{\alpha_{g_i}}(x)$ are calculated in Appendices B and D, assuming M-ary PAM transmission.

Note that typically, $\mathcal{I}_f$ is unknown, and the pdf of the s^{th} smallest β is

$$f_{\beta_s}(x) = \sum_{\forall \mathcal{I}_f} \Pr(\mathcal{I}_f)\, f_{\beta_s}(x|\mathcal{I}_f) \ , \tag{27}$$

where the summation is over all $\binom{N}{b}$ distinct $\mathcal{I}_f$. We may lower bound the pdf by considering the first few terms only of the summation, however it is found that a sufficiently accurate approximation results from considering the most likely of all sets $\mathcal{I}_f$, that is the set corresponding to the indices with the highest probability of error, so that

$$f_{\beta_s}(x) \simeq \arg\max_{\Pr(\mathcal{I}_f)} f_{\beta_s}(x|\mathcal{I}_f) \ \text{ and } f_{\gamma_s}(x) \simeq \arg\max_{\Pr(\mathcal{I}_g)} f_{\gamma_s}(x|\mathcal{I}_g) \ . \tag{28}$$

While the pdfs of the order statistics in (25),(26) are elegant expressions, they are difficult to calculate, since the calculation of the permanent of an n-by-n matrix requires on the order of $n2^n$ calculations. For large matrices, say $n > 30$ obtaining the permanent is not feasible. Since we use the permanent expressions in the calculation of a lower bound, (22), we therefore seek to bound the permanent of the matrices in (25) and (26).

We exploit the fact that the matrices in (25) and (26) are non-negative to apply the bounds of [21], in [20], to obtain

$$\begin{aligned} \text{Per}|A| &\geq \prod_{i=1}^{n} \sum_{t=1}^{i} a'_{it} + (r_1 - n\, a'_{11}) \prod_{j=2}^{n} \sum_{s=1}^{j-1} a'_{js} \ , \text{ and} \\ \text{Per}|A| &\leq \prod_{i=1}^{n} \sum_{t=1}^{i} a^*_{it} + (r_1 - n\, a^*_{11}) \prod_{j=2}^{n} \sum_{s=1}^{j-1} a'_{js} \ , \end{aligned} \tag{29}$$

where A is an n-by-n matrix, with elements a_{ij}. r_1 denotes the first row sum of A, the i^{th} row is denoted $A_{(i)} = (a_{i1}, a_{i2}, \ldots, a_{in})$, and $(a'_{i1}, a'_{i2}, \ldots, a'_{in})$ is the n-tuple representing the i^{th} row elements arranged in nondecreasing order, $a'_{i1} \leq a'_{i2} \leq \ldots \leq a'_{in}$. Similarly, $(a^*_{i1}, a^*_{i2}, \ldots, a^*_{in})$ is the n-tuple representing the i^{th} row elements arranged in nonincreasing order, $a^*_{i1} \geq a^*_{i2} \geq \ldots \geq a^*_{in}$. We can readily apply these lower and upper bounds to the permanent expressions of (25) and (26) to obtain lower and upper bounds on the probability density functions $f_{\beta_s}(x)$ and $f_{\gamma_s}(x)$. Using these bounds we readily obtain a lower bound on $\Pr(\mathcal{S}_{b,k})$ from (24). Consequently, a lower bound on $\Pr(\mathcal{S}_b)$, (22), and an upper bound on $\Pr(\mathcal{F})$, (21), are obtained.

For the special case of the AWGN channel, obtaining $f_{\beta_s}(x)$ and $f_{\gamma_s}(x)$ is straightforward. All β statistics are iid, with cdf $F_\beta(x)$ and pdf $f_\beta(x)$. Similarly all γ statistics are iid with pdf $f_\gamma(x)$ and cdf $F_\gamma(x)$. Using a basic result of order statistics [22] we readily obtain

$$\begin{aligned} f_{\beta_s}(x) =& \frac{b!}{(s-1)!(b-s)!} \left[F_\beta(x)\right]^{s-1} \left[1-F_\beta(x)\right]^{b-s} f_\beta(x) \ , \\ f_{\gamma_s}(x) =& \frac{(N-b)!}{(s-1)!(n-b-s)!} \left[F_\gamma(x)\right]^{s-1} \left[1-F_\gamma(x)\right]^{N-b-s} f_\gamma(x) \ . \end{aligned} \tag{30}$$

Note that (30) may be obtained from (25) for the special case of all γ_i being iid, and all β_i being iid.

4 .3 Multistage Performance

Given an m-level construction C lattice we now calculate the probability of lattice decoding error for multistage GMD decoding. A decoding error occurs if the estimated lattice point, obtained by summing the GMD decoding stage output, is not equal to the transmitted lattice point. We denote this event $\mathcal{E}_\Lambda$, and denote the events of correct and incorrect decoding at the l^{th} stage as $\mathcal{E}_l^c$ and $\mathcal{E}_l$ respectively , such that $\mathcal{E}_\Lambda = \cup_{l=1}^m \mathcal{E}_l$. Using (21), for each stage we may approximate the probability of error conditional on all previous stages being correctly decoded, that is $\Pr(\mathcal{F}) \approx \Pr\left(\mathcal{E}_l | [\mathcal{E}_{l-1}^c \cap \mathcal{E}_{l-2}^c \cap \ldots \cap \mathcal{E}_1^c]\right)$. We may then upper bound the lattice error rate as the probability that at least one decoding stage is in error

$$\Pr(\mathcal{E}_\Lambda) = \Pr\left(\bigcup_{l=1}^m \mathcal{E}_l\right) \leq \sum_{l=1}^m \Pr\left(\mathcal{E}_l | \left\{\mathcal{E}_{l-1}^c \cap \mathcal{E}_{l-2}^c \cap \ldots \cap \mathcal{E}_1^c\right\}\right) \ . \tag{31}$$

Assuming the probability of error at each decoding stage is small, this upper bound also approximates the probability of lattice decoding error.

We have thus extended the analysis of [4] to calculate an approximation to the performance of GMD decoding construction C lattices for QAM based OFDM systems transmitting over frequency selective channels. The approximation appears to be very good, and is observed to be a good upper bound for many cases of interest. This is demonstrated through simulation in the next Section.

5 Simulations

We compare the calculated analytical approximations of GMD decoding error rates with simulated system error rates. We consider a 64 subcarrier OFDM system occupying 30MHz total bandwidth, with each subcarrier transmitting a 256-QAM constellation. Each OFDM block is mapped from a point in the 128 dimensional Barnes-Wall lattice, as described previously. At each stage we perform GMD decoding to obtain an estimate of the transmitted lattice point.

We consider the lattice point, or equivalently OFDM block, error rate for the AWGN and three randomly generated frequency selective channels. In all cases we assume perfect channel state information, time synchronisation and frequency synchronisation. Channels A and B have an exponential power delay profile with mean excess delay of 50ns, while channel C is a Rician channel with similar diffuse component but a 10dB Rice factor. The channel frequency responses are shown in Figure 1.

The simulated error rates and analytical approximations for the AWGN channel and channel A are shown in Figure 2. We also plot the block error rate for an uncoded 64 subcarrier OFDM system, transmitting 256-QAM subcarriers on both an AWGN channel and on the frequency selective channel A. Similar results for channels B and C are displayed in Figure 3.

We observe that the analysis provides good upper bounds, with accuracy within 1dB , 0.5dB, 2dB and 0.5dB at an error rate of 10^{-5} for the AWGN channel and channels A, B, and C respectively. In addition, note the ablility to calculate the approximate upper bounds to arbitrarily small error rates: error rates of 10^{-8} are shown, whereas accurate simulation of the system at these error rates is not generally feasible.

The simulations and analysis both demonstrate the large coding gains provided by lattice encoding the OFDM symbol block. For example, we estimate a coding gain of approximately 9dB and 14dB at an error rate of 10^{-6}, for transmission across the AWGN channel and channel A respectively. Such large gains are due to the properties of the 128-dimensional Barnes-Wall lattice.

6 Concluding Remarks

We have derived an approximation of the error rate of multistage GMD decoding of lattice encoded OFDM systems, which are tight upper bounds in the observed cases of interest. We address the use of lattices to encode OFDM blocks, to provide high coding gains without the latency of encoding data over successive OFDM blocks. Furthermore, we show that lattice encoding of OFDM systems with high dimensional lattices can provide excellent coding gains with low complexity decoding.

The approximations obtained are valid for OFDM transmission over frequency selective channels, conditions ubiquitous in the wireless environment. They are derived for an arbitrary frequency selective channel, and an arbitrary construction C lattice mapped to any N , M^2-ary QAM OFDM subcarriers. The approximations are derived with probabilistic analysis, using recent advances in order statistics distributions, and combinatorial mathematics, as well as expressions obtained for the probability distributons of GMD reliability statistics. However, calculation of the approximation is simple, requiring low computational complexity.

This analysis is useful in enabling system designers and operators to rapidly calculate best obtainable error rates. Such information may be used to control, and predict the performance, of OFDM, where parameters such as throughput, power requirements and number of subcarriers may be adapted.

Appendix A: Reliability Density Function Preliminaries

Given an M-ary PAM constellation, separated uniformly by $2\sqrt{E_0}$, we label the constellation points as p^0, $p^1, \ldots, p^{M-1}$. Therefore $p^k = \sqrt{E_0}(2k - M + 1)$, for $k = 0, 1, \ldots, M-1$. We label transmitted PAM points as p^s, for $s = 0, 1, \ldots, M-1$, and we receive a noise corrupted version, $p^s + n$ of this point, where n is additive white Gaussian noise with mean zero and one-dimensional variance σ^2. At the receiver we make a hard decision estimate of the transmitted point, by choosing the closest constellation point to the received noise corrupted point, we denote this decision p^r.

With each hard decision we associate a reliability metric, α, defined as a function of the distance between the received point, $p^s + n$, and the hard decision point. Specifically

$$\begin{aligned} \alpha &= \left| 1 - \frac{d}{\sqrt{E_0}} \right| \ , \\ \text{with } d &= |p^r - (p^s + n)| \ . \end{aligned} \tag{32}$$

The additive white Gaussian noise (AWGN) component, n, a Gaussian distribution with mean zero, and variance σ^2. It may be seen that $\alpha \geq 0$, and in the case of $p^r : r \in \{1, 2, \ldots, M-2\}$; that is, ignoring constellation end points, $0 \leq \alpha \leq 1$.

If the hard decision made is correct, that is $p^r \equiv p^s$, so $r = s$, then we denote the reliability α associated with this correct hard decision as γ. If the hard decision is incorrect, that is $p^r \not\equiv p^s$, so $r \neq s$ then we denote the reliability α associated with this incorrect hard decision as β. We wish to find exact expressions for the probability density function (pdf) and cumulative probability density function (cdf) of reliability statistics γ and β, in both cases for arbitrary M, E_0 and σ^2.

Before we proceed with the PDF and CDF derivations, make note of the following useful expressions and definitions. The Gaussian probability density function is

$$\begin{aligned} f(x) =& \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2} \left[\frac{x-\mu}{\sigma} \right]^2 \right) \\ \text{with } \int_0^z f(x)\, dx =& \frac{1}{2} \text{erf}\left(\frac{z}{\sqrt{2}\sigma} \right) \ , \end{aligned} \tag{33}$$

for mean μ and variance σ^2. The exponential integral, or error functions have properties

$$\begin{aligned}
\mathrm{erf}\,(z) &= \frac{2}{\sqrt{\pi}} \int_0^z \exp^{-t^2}\, dt \\
\mathrm{erfc}\,(z) &= 1 - \mathrm{erf}\,(z) \\
\mathrm{erf}\,(\infty) &= 1, \qquad \mathrm{erf}\,(0) = 0, \qquad -\mathrm{erf}\,(z) = \mathrm{erf}\,(-z) \\
\int_L^U \exp\left(-A[x-B]^2\right)\, dx &= \frac{\sqrt{\pi}}{2\sqrt{A}} \left[\mathrm{erf}\left(\sqrt{A}[U-B]\right) - \mathrm{erf}\left(\sqrt{A}[L-B]\right)\right]
\end{aligned} \tag{34}$$

Appendix B: Probability Density Function of γ

If the hard decision is correct, then $p^r = p^s$ and therefore $r = s$. Therefore $d = |n|$, and we let $\gamma' = \left|1 - \frac{|n|}{\sqrt{E_0}}\right|$ be the *unclipped reliability*, so that $\gamma = \min\{1, \gamma'\}$. Furthermore, we may express the total probability density function of γ as

$$f_\gamma(x) = \sum_{s=0}^{M-1} \Pr(s = k)\, f_\gamma(x|r = s = k) = \sum_{s=0}^{M-1} \frac{1}{M} f_\gamma(x|r = s = k) \ , \tag{35}$$

that is the sum of conditional probabilities given the transmitted symbol p^s, and assuming equiprobable transmission of all symbols. We therefore find each of the conditional probability density functions, as follows.

For $s = 1, 2, \ldots, M-2$, the assumption that $p^r = p^s$ implies that $-\sqrt{E_0} \leq n \leq \sqrt{E_0}$, and therefore n has conditional pdf

$$f_n(x|r = 1, 2, \ldots M-2) = \begin{cases} \frac{1}{\mathrm{erf}\,\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x}{\sigma}\right]^2\right) & \text{for } -\sqrt{E_0} \leq x \leq \sqrt{E_0} \\ 0 & \text{otherwise;} \end{cases} \tag{36}$$

that is, a two sided truncated Gaussian PDF. Therefore, the magnitude of the noise component, $d = |n|$, has PDF

$$f_d(x|r = 1, 2, \ldots M-2) = \begin{cases} \frac{2}{\mathrm{erf}\,\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x}{\sigma}\right]^2\right) & \text{for } 0 \leq x \leq \sqrt{E_0} \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Now since $0 < d < \sqrt{E_0}$, therefore $\gamma' = 1 - \frac{d}{\sqrt{E_0}}$, so using the transformation technique for functions of random variables

$$\begin{aligned} f_{\gamma'}(x|r = 1, 2, \ldots M-2) &= \left|\frac{\partial(d)}{\partial(\gamma)}\right| f_d([\sqrt{E_0} - \sqrt{E_0}\gamma]|0 \leq x \leq \sqrt{E_0}) \\ &= \sqrt{E_0} f_d([\sqrt{E_0} - \sqrt{E_0}\gamma]|0 \leq x \leq \sqrt{E_0}) \\ &= \begin{cases} \frac{2}{\mathrm{erf}\,\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2}[1-x]^2\right) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{38}$$

For $r = 0$, the assumption that $p^r = p^s = p^0$ implies that $n \leq \sqrt{E_0}$, so that the noise has the conditional PDF

$$f_n(x|r=0) = \begin{cases} \frac{1}{\frac{1}{2}+\frac{1}{2}\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x}{\sigma}\right]^2\right) & \text{for } x \leq \sqrt{E_0} \\ 0 & \text{otherwise,} \end{cases} \tag{39}$$

that is, a one-sided truncated Gaussian PDF. Therefore, the distance, $d = |n|$, from the received point to p^0, has PDF

$$f_d(x|r=0) = \begin{cases} \frac{4}{1+\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x}{\sigma}\right]^2\right) & \text{for } 0 \leq x \leq \sqrt{E_0} \\ \frac{2}{1+\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x}{\sigma}\right]^2\right) & \text{for } x > \sqrt{E_0} \\ 0 & \text{otherwise.} \end{cases} \tag{40}$$

Observe that the PDF of $1 - \frac{d}{\sqrt{E_0}}$ is

$$f_{1-\frac{d}{\sqrt{E_0}}}(x|r=0) = \begin{cases} \frac{4\sqrt{E_0}}{1+\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2}\left[x-1\right]^2\right) & \text{for } 0 \leq x \leq 1 \\ \frac{2\sqrt{E_0}}{1+\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2}\left[x-1\right]^2\right) & \text{for } x < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{41}$$

Therefore, the conditional pdf of $\gamma' = \left|1 - \frac{d}{\sqrt{E_0}}\right|$ is found to be

$$\begin{aligned} f_{\gamma'}(x|r=0) &= \\ &\begin{cases} \frac{4\sqrt{E_0}}{1+\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2}\left[x-1\right]^2\right) + \frac{2\sqrt{E_0}}{1+\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2}\left[-x-1\right]^2\right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \leq x \leq 1 & \\ \frac{2\sqrt{E_0}}{1+\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2}\left[-x-1\right]^2\right) \qquad \text{for } x > 1 & \\ 0 \qquad\qquad\qquad\qquad \text{otherwise.} & \end{cases} \\ &= \begin{cases} \frac{2\sqrt{E_0}}{1+\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \left[2\exp\left(-\frac{E_0}{2\sigma^2}\left[x-1\right]^2\right) + \exp\left(-\frac{E_0}{2\sigma^2}\left[x+1\right]^2\right)\right] & \text{for } 0 \leq x \leq 1 \\ \frac{2\sqrt{E_0}}{1+\mathrm{erf}\frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2}\left[x+1\right]^2\right) & \text{for } x > 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{42}$$

It is readily observed that the PDF of γ', conditional on point p^{M-1} is identical to the above PDF. That is, $f_{\gamma'}(x|r=0) = f_{\gamma}(x|r=M-1)$. Therefore, combining the above conditional PDFs for γ', we may write

$$f_{\gamma'}(x) = \frac{1}{M}\sum_{s=0}^{M-1} f_{\gamma'}(x|r=s)$$
$$= \begin{cases} \frac{M-2}{M}\frac{2}{\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{E_0}{2\sigma^2}[1-x]^2\right) & \\ \qquad +\frac{2}{M}\frac{2\sqrt{E_0}}{1+\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{1}{\sqrt{2\pi}\sigma}\left[2\exp\left(-\frac{E_0}{2\sigma^2}[x-1]^2\right)+\exp\left(-\frac{E_0}{2\sigma^2}[x+1]^2\right)\right] & \text{for } 0 \le x \le 1 \\ \frac{1}{M}\frac{4\sqrt{E_0}}{1+\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{E_0}{2\sigma^2}[x+1]^2\right) & \text{for } x > 1 \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

Therefore the pdf of the reliability associated with a correct hard decision, $\gamma = \min\{1, \gamma'\}$, is

$$f_{\gamma}(x)$$
$$= \begin{cases} \frac{M-2}{M}\frac{2}{\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{E_0}{2\sigma^2}[1-x]^2\right) & \\ \qquad +\frac{2}{M}\frac{2\sqrt{E_0}}{1+\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{1}{\sqrt{2\pi}\sigma}\left[2\exp\left(-\frac{E_0}{2\sigma^2}[x-1]^2\right)+\exp\left(-\frac{E_0}{2\sigma^2}[x+1]^2\right)\right] & \text{for } 0 \le x < 1 \\ \frac{M-2}{M}\frac{2}{\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} + \frac{2}{M}\frac{2}{1+\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma}\left[2+\exp\left(-\frac{2E_0}{\sigma^2}\right)\right] & \\ \qquad +\int_1^\infty \frac{1}{M}\frac{4\sqrt{E_0}}{1+\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{E_0}{2\sigma^2}[x+1]^2\right)dx & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} \frac{M-2}{M}\frac{2}{\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma}\exp\left(-\frac{E_0}{2\sigma^2}[1-x]^2\right) & \\ \qquad +\frac{2}{M}\frac{2\sqrt{E_0}}{1+\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{1}{\sqrt{2\pi}\sigma}\left[2\exp\left(-\frac{E_0}{2\sigma^2}[x-1]^2\right)+\exp\left(-\frac{E_0}{2\sigma^2}[x+1]^2\right)\right] & \text{for } 0 \le x < 1 \\ \frac{1}{M}\left\{\frac{2[M-2]}{\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} + \frac{4}{1+\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma}\left[2+\exp\left(-\frac{2E_0}{\sigma^2}\right)\right] + \frac{2}{1+\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}}\right\} & \text{for } x = 1 \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

Appendix C: Probability Density Function of β

We now derive the pdf of a reliability statistic associated with an incorrect hard decision, that is α, given that $p^r \not\equiv p^s$. We denote this statistic $\beta = \min\{1, \beta'\}$, where $\beta' = \left|1 - \frac{d}{\sqrt{E_0}}\right|$ is the *unclipped reliability*, calculated from the distance d between the received point and the hard decision point. We may express the pdf of β' as the sum of conditional reliabilities

$$\begin{aligned} f_{\beta'}(x) &= \sum_{m=0}^{M-1} \frac{1}{1 - \Pr(r = s|s = m)} \sum_{l=0, l\neq s}^{M-1} \Pr(s = m \cap r = l) f_\beta(x|s = m, r = l) \ , \\ &= \sum_{m=0}^{M-1} \frac{1}{1 - \Pr(r = s|s = m)} \sum_{l=0, l\neq s}^{M-1} \Pr(s = m) \Pr(r = l|s = m) f_\beta(x|s = m, r = l) \ , \\ &= \frac{1}{M} \sum_{m=0}^{M-1} \frac{1}{1 - \Pr(r = s|s = m)} \sum_{l=0, l\neq s}^{M-1} \Pr(r = l|s = m) f_\beta(x|s = m, r = l) \ ; \end{aligned} \tag{45}$$

assuming equiprobable transmission of all PAM constellation points. We firstly require the probability of correct hard decision given that point s is transmitted. For $s = 0$, a correct hard decision implies that $n \leq \sqrt{E_0}$, hence $\Pr(r = s|s = 0) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)$. Similarly, $\Pr(r = s|s = M - 1) = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)$. For $s = 1, 2, \ldots, M - 2$ we find the correct hard decision decoding implies that $-\sqrt{E_0} \leq n \leq \sqrt{E_0}$ and therefore $\Pr(r = s|s = 1, 2, \ldots, M - 2) = \mathrm{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)$.

We now calculate the conditional probability density functions of β', given that the hard decision is p^0, and point p^m was transmitted. The noise must then satisfy $n \leq -\sqrt{E_0}[2m - 1]$. Therefore the noise has pdf

$$f_n(x|s = m, r = 0) = \begin{cases} \frac{1}{\Pr(r=0|s=m)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x}{\sigma}\right]^2\right) & \text{for } x \leq -\sqrt{E_0}(2m - 1) \\ 0 & \text{otherwise ;} \end{cases} \tag{46}$$

that is, a one-sided Gaussian tail distribution. The signed distance, $d' = n + 2\sqrt{E_0}m$ to the point $r = 0$ therefore has PDF

$$f_{d'}(x|s = m, r = 0) = \begin{cases} \frac{1}{\Pr(r=0|s=m)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x-2\sqrt{E_0}m}{\sigma}\right]^2\right) & \text{for } x \leq \sqrt{E_0} \\ 0 & \text{otherwise ;} \end{cases} \tag{47}$$

and the distance, $d = |d'| = |n + 2\sqrt{E_0}m|$ to the hard decision point p^0 has PDF

$$f_d(x|s=m, r=0) = \begin{cases} \frac{1}{\Pr(r=0|s=m)} \frac{1}{\sqrt{2\pi}\sigma} \left[\exp\left(-\frac{1}{2} \left[\frac{x - 2\sqrt{E_0}m}{\sigma} \right]^2 \right) + \exp\left(-\frac{1}{2} \left[\frac{x + 2\sqrt{E_0}m}{\sigma} \right]^2 \right) \right] \\ \qquad \text{for } 0 \le x \le \sqrt{E_0} \\ \frac{1}{\Pr(r=0|s=m)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2} \left[\frac{x + 2\sqrt{E_0}m}{\sigma} \right]^2 \right) \\ \qquad \text{for } x > \sqrt{E_0} \\ 0 \qquad \text{otherwise .} \end{cases} \tag{48}$$

We find the PDF of $1 - \frac{d}{\sqrt{E_0}}$ to be

$$\begin{aligned} & f_{1-\frac{d}{\sqrt{E_0}}}(x|s=m, r=0) \\ &= \begin{cases} \frac{1}{\Pr(r=0|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \left[\exp\left(-\frac{E_0}{2\sigma^2} \left[1 - x - 2m\right]^2 \right) + \exp\left(-\frac{E_0}{2\sigma^2} \left[1 - x + 2m\right]^2 \right) \right] & \text{for } 0 \le x \le 1 \\ \frac{1}{\Pr(r=0|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2} \left[1 - x + 2m\right]^2 \right) & \text{for } x < 0 \\ 0 & \text{otherwise .} \end{cases} \end{aligned} \tag{49}$$

and therefore the reliability, $\beta' = \left| 1 - \frac{d}{\sqrt{E_0}} \right|$, has PDF

$$\begin{aligned} & f_{\beta'}(x|s=m, r=0) \\ &= \begin{cases} \frac{1}{\Pr(r=0|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \left[\exp\left(-\frac{E_0}{2\sigma^2} \left[1 - x - 2m\right]^2 \right) + \exp\left(-\frac{E_0}{2\sigma^2} \left[1 - x + 2m\right]^2 \right) \right. \\ \left. + \exp\left(-\frac{E_0}{2\sigma^2} \left[1 + x + 2m\right]^2 \right) \right] \text{ for } 0 \le x \le 1 \\ \frac{1}{\Pr(r=0|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2} \left[1 + x + 2m\right]^2 \right) \text{ for } x > 1 \\ 0 \text{ otherwise .} \end{cases} \end{aligned} \tag{50}$$

For the case of $r = M - 1$, given that the point p^s is transmitted, we deduce that the noise must satisfy $n \ge \sqrt{E_0}[2(M-1-s) - 1] = \sqrt{E_0}[2M - 2m - 3]$. Therefore the noise has PDF

$$f_n(x|s=m, r=M-1) = \begin{cases} \frac{1}{\Pr(r=M-1|s=m)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2} \left[\frac{x}{\sigma} \right]^2 \right) & \text{for } x > -\sqrt{E_0}[2M - 2m - 3] \\ 0 & \text{otherwise ;} \end{cases} \tag{51}$$

that is, a one-sided Gaussian tail distribution. The signed distance, d' to the point $r = M-1$ therefore given as $d' = n - 2\sqrt{E_0}[r-m] = n - 2\sqrt{E_0}[M-1-m]$, and therefore has PDF

$$f_{d'}(x|s=m, r=M-1) = \begin{cases} \frac{1}{\Pr(r=M-1|s=m)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x+2\sqrt{E_0}[M-1-m]}{\sigma}\right]^2\right) & \text{for } x \geq -\sqrt{E_0} \\ 0 & \text{otherwise} . \end{cases} \tag{52}$$

Therefore the pdf of the distance, $d = |d'|$ to the hard decision point p^{M-1} is

$$\begin{aligned} & f_d(x|s=m, r=M-1) \\ &= \begin{cases} \frac{1}{\Pr(r=0|s=M-1)} \frac{1}{\sqrt{2\pi}\sigma} \left[\exp\left(-\frac{1}{2}\left[\frac{x+2\sqrt{E_0}[M-1-m]}{\sigma}\right]^2\right) + \exp\left(-\frac{1}{2}\left[\frac{x-2\sqrt{E_0}[M-1-m]}{\sigma}\right]^2\right)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \geq x \geq \sqrt{E_0} \\ \frac{1}{\Pr(r=0|s=M-1)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x+2\sqrt{E_0}[M-1-m]}{\sigma}\right]^2\right) \qquad \text{for } x > \sqrt{E_0} \\ 0 \qquad \text{otherwise} . \end{cases} \end{aligned} \tag{53}$$

and the pdf of $1 - \frac{d}{\sqrt{E_0}}$ is found to be

$$\begin{aligned} & f_{1-\frac{d}{\sqrt{E_0}}}(x|s=m, r=M-1) \\ &= \begin{cases} \frac{1}{\Pr(r=M-1|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \left[\exp\left(-\frac{E_0}{2\sigma^2}\left[1-x+2[M-1-m]\right]^2\right) + \exp\left(-\frac{E_0}{2\sigma^2}\left[1-x-2[M-1-m]\right]^2\right)\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } 0 \geq x \geq 1 \\ \frac{1}{\Pr(r=M-1|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2}\left[1-x+2[M-1-m]\right]^2\right) \qquad \text{for } x < 0 \\ 0 \qquad \text{otherwise} . \end{cases} \end{aligned} \tag{54}$$

We then show the conditional pdf of $\beta' = \left|1 - \frac{d}{\sqrt{E_0}}\right|$ to be

$$
\begin{aligned}
& f_{\beta'}(x|s=m, r=M-1) \\
&= \begin{cases}
\frac{1}{\Pr(r=M-1|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \Big[\exp\left(-\frac{E_0}{2\sigma^2}\left[1-x+2[M-1-m]\right]^2\right) + \exp\left(-\frac{E_0}{2\sigma^2}\left[1-x-2[M-1-m]\right]^2\right) \\
\qquad + \exp\left(-\frac{E_0}{2\sigma^2}\left[1+x+2[M-1-m]\right]^2\right)\Big] & \text{for } 0 \geq x \geq 1 \\
\frac{1}{\Pr(r=M-1|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \exp\left(-\frac{E_0}{2\sigma^2}\left[1+x+2[M-1-m]\right]^2\right) & \text{for } x > 1 \\
0 & \text{otherwise .}
\end{cases}
\end{aligned} \tag{55}
$$

Finally, for hard decision points p^r for some $r = \ell \in \{1, 2, \ldots, M-2\}$, given that the point $p^{s=m}$ is sent we find that the noise satisfies $2\sqrt{E_0}[r-m] - \sqrt{E_0} \geq n \geq 2\sqrt{E_0}[r-m] + \sqrt{E_0}$. Therefore the noise has pdf

$$
\begin{aligned}
& f_n(x|s=m, r=\ell) \\
&= \begin{cases}
\frac{1}{\Pr(r=\ell|s=m)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x}{\sigma}\right]^2\right) & \text{for } 2\sqrt{E_0}[\ell-m] - \sqrt{E_0} \geq x \geq 2\sqrt{E_0}[\ell-m] + \sqrt{E_0} \\
0 & \text{otherwise ;}
\end{cases}
\end{aligned} \tag{56}
$$

that is a segment of a Gaussian pdf. The signed distance, d' to the nearest point is given by $d' = n - 2\sqrt{E_0}[r-m]$, such that this signed distance has PDF

$$
\begin{aligned}
& f_{d'}(x|s=m, r=\ell) \\
&= \begin{cases}
\frac{1}{\Pr(r=\ell|s=m)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left[\frac{x+2\sqrt{E_0}[\ell-m]}{\sigma}\right]^2\right) & \text{for } -\sqrt{E_0} \geq x \geq \sqrt{E_0} \\
0 & \text{otherwise .}
\end{cases}
\end{aligned} \tag{57}
$$

The distance magnitude, $d = |d'|$, to the hard decision point p^r therefore has pdf

$$
\begin{aligned}
& f_d(x|s=m, r=\ell) \\
&= \begin{cases}
\frac{1}{\Pr(r=\ell|s=M-1)} \frac{1}{\sqrt{2\pi}\sigma} \left[\exp\left(-\frac{1}{2}\left[\frac{x+2\sqrt{E_0}[\ell-m]}{\sigma}\right]^2\right) + \exp\left(-\frac{1}{2}\left[\frac{x-2\sqrt{E_0}[\ell-m]}{\sigma}\right]^2\right)\right] & \text{for } 0 \geq x \geq \sqrt{E_0} \\
0 & \text{otherwise .}
\end{cases}
\end{aligned} \tag{58}
$$

Consequently, we find the reliability $\beta' = \left|1 - \frac{d}{\sqrt{E_0}}\right| = 1 - \frac{d}{\sqrt{E_0}}$ has PDF

$$
\begin{aligned}
&f_{\beta'}(x|r=\ell, s=m) \\
&= \begin{cases} \frac{1}{\Pr(r=\ell|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \left[\exp\left(-\frac{E_0}{2\sigma^2}\left[1-x+2(\ell-m)\right]^2\right) + \exp\left(-\frac{E_0}{2\sigma^2}\left[1-x-2(\ell-m)\right]^2\right)\right] & \text{for } 0 \geq x \geq 1 \\ 0 & \text{otherwise .} \end{cases}
\end{aligned}
\tag{59}
$$

Therefore, we may combine the above conditional PDFs for β' to obtain the pdf of β'. After arduous expansions and simplifications we obtain

$$
\begin{aligned}
f_{\beta'}(x) =& \frac{1}{M}\sum_{m=0}^{M-1} \frac{1}{1-\Pr(r=s|s=m)} \sum_{\ell=0,\ell\neq m}^{M-1} \Pr(r=\ell|s=m)\, f_{\beta}(x|r=\ell, s=m) \\
=& \begin{cases} \frac{\sqrt{E_0}}{M\sqrt{2\pi}\sigma}\left\{\sum_{m=1}^{M-1} \frac{2}{1-\Pr(r=\ell|s=m)} \exp\left(-\frac{E_0}{2\sigma^2}[x+2m+1]^2\right)\right. \\ \qquad \left. + \sum_{m=0}^{M-1}\sum_{\ell=0,\ell\neq m}^{M-1} \frac{2}{1-\Pr(r=\ell|s=m)} \exp\left(-\frac{E_0}{2\sigma^2}[x+2(m-\ell)-1]^2\right)\right\} & \text{for } 0 \geq x \geq 1 \\ \frac{\sqrt{E_0}}{M\sqrt{2\pi}\sigma}\left\{\frac{2}{1-\Pr(r=0|s=0)} \exp\left(-\frac{E_0}{2\sigma^2}[x+2(M-1)+1]^2\right) + \right. \\ \qquad \left. \frac{1}{1-\Pr(r=m|s=m)} \sum_{m=1}^{M-2} \exp\left(-\frac{E_0}{2\sigma^2}[x+2m+1]^2\right)\right\} & \text{for } x > 1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
\tag{60}
$$

Where we recall that

$$
\Pr(r=s|s=m) \begin{cases} \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right) & \text{for } m = 1, 2, \ldots, M-2 \\ \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right) & \text{for } m = 0, M-1 \end{cases}
\tag{61}
$$

Therefore, from (60), we can calculate the pdf of $\beta = \min\{1, \beta'\}$ as

$$
\begin{aligned}
& f_\beta(x) \\
&= \begin{cases}
\frac{\sqrt{E_0}}{M\sqrt{2\pi}\sigma} \Big\{ \sum_{m=1}^{M-1} \frac{2}{1-\Pr(r=\ell|s=m)} \exp\left(-\frac{E_0}{2\sigma^2}[x+2m+1]^2\right) & \\
\qquad + \sum_{m=0}^{M-1} \sum_{\ell=0,\ell\neq m}^{M-1} \frac{2}{1-\Pr(r=\ell|s=m)} \exp\left(-\frac{E_0}{2\sigma^2}[x+2(m-\ell)-1]^2\right) \Big\} & \text{for } 0<x<1 \\
\int_1^\infty \frac{\sqrt{E_0}}{M\sqrt{2\pi}\sigma} \Big\{ \frac{2}{1-\Pr(r=0|s=0)} \exp\left(-\frac{E_0}{2\sigma^2}[x+2(M-1)+1]^2\right) + & \\
\qquad \frac{1}{1-\Pr(r=m|s=m)} \sum_{m=1}^{M-2} \exp\left(-\frac{E_0}{2\sigma^2}[x+2m+1]^2\right) \Big\} dx + f_{\beta'}(x)\Big|_{x=1} & \text{for } x=1 \\
0 & \text{otherwise.}
\end{cases} \\
&= \begin{cases}
\frac{\sqrt{E_0}}{M\sqrt{2\pi}\sigma} \Big\{ \sum_{m=1}^{M-1} \frac{2}{1-\Pr(r=\ell|s=m)} \exp\left(-\frac{E_0}{2\sigma^2}[x+2m+1]^2\right) & \\
\qquad + \sum_{m=0}^{M-1} \sum_{\ell=0,\ell\neq m}^{M-1} \frac{2}{1-\Pr(r=\ell|s=m)} \exp\left(-\frac{E_0}{2\sigma^2}[x+2(m-\ell)-1]^2\right) \Big\} & \text{for } 0<x<1 \\
\frac{2\sqrt{E_0}}{M\sqrt{2\pi}\sigma} \Big\{ \sum_{m=1}^{M-1} \frac{1}{1-\Pr(r=\ell|s=m)} \exp\left(-\frac{E_0}{2\sigma^2}[2+2m]^2\right) & \\
\qquad + \sum_{m=0}^{M-1} \sum_{\ell=0,\ell\neq m}^{M-1} \frac{1}{1-\Pr(r=\ell|s=m)} \exp\left(-\frac{E_0}{2\sigma^2}[2(m-\ell)]^2\right) \Big\} & \\
\qquad + \frac{1}{2M} \Big\{ \frac{2}{1-\Pr(r=0|s=0)} \mathrm{erfc}\left(M\sqrt{\frac{2E_0}{\sigma^2}}\right) + \frac{1}{1-\Pr(r=m|s=m)} \sum_{m=1}^{M-2} \mathrm{erfc}\left(\sqrt{\frac{2E_0}{\sigma^2}}[1+m]\right) \Big\} & \\
& \text{for } x=1 \\
0 & \text{otherwise.}
\end{cases}
\end{aligned} \tag{62}
$$

Appendix D: Cumulative Probability Density Function of γ

Note that the CDF of γ may be expressed as

$$F_\gamma(x) = \int_{-\infty}^{x} f_\gamma(y)\, dy = \begin{cases} \int_0^x f_{\gamma'}(y)\, dy & \text{for } 0 \le x < 1 \\ 1 & \text{for } x \ge 1 \\ 0 & \text{otherwise.} \end{cases} \tag{63}$$

Therefore we take the derivative of (44) to find the CDF of γ. We first consider the case where $0 \le x < 1$, and we find that

$$\begin{aligned}
&F_\gamma(x)\Big|_{0\le x<1} \\
&= \int_0^x f_\gamma(y)\Big|_{0\le y<1} dy \\
&= \frac{M-2}{M} \frac{2\sqrt{E_0}}{\sqrt{2\pi}\sigma \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)} \int_0^x \exp\left(-\frac{E_0}{2\sigma^2}[y-1]^2\right) dy \\
&\qquad + \frac{2}{M} \frac{2\sqrt{E_0}}{1+\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)} \frac{1}{\sqrt{2\pi}\sigma} \left[2\int_0^x \exp\left(-\frac{E_0}{2\sigma^2}[y-1]^2\right) dy + \int_0^x \exp\left(-\frac{E_0}{2\sigma^2}[y+1]^2\right) dy\right] \\
&= \frac{M-2}{M} \frac{2\sqrt{E_0}}{\sqrt{2\pi}\sigma \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)} \frac{\sqrt{\pi}}{2\sqrt{\frac{E_0}{2\sigma^2}}} \left[\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1]\right) - \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0-1]\right)\right] \\
&\qquad + \frac{2}{M} \frac{2\sqrt{E_0}}{1+\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)} \frac{1}{\sqrt{2\pi}\sigma} \frac{\sqrt{\pi}}{2\sqrt{\frac{E_0}{2\sigma^2}}} \left[2\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1]\right) - 2\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0-1]\right)\right. \\
&\qquad \left. + \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x+1]\right) - \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0+1]\right)\right],
\end{aligned} \tag{64}$$

where we have applied the exponential integral result of (34). Therefore we may write

$$\begin{aligned}
F_\gamma(x)\Big|_{0\le x<1} =& \frac{M-2}{M} \frac{1}{\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)} \left[\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1]\right) + \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)\right] \\
&+ \frac{2}{M} \frac{1}{1+\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)} \left[2\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1]\right) + \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x+1]\right) + \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)\right].
\end{aligned} \tag{65}$$

For the remaining piecewise component, that is, for $x \ge 1$ it is easily shown that $F_\beta(x)\Big|_{x\ge 1} = 1$. Combining the two piecewise components of the expression for the CDF of γ we obtain the desired final result

$$F_\gamma(x) = \begin{cases} \frac{M-2}{M} \frac{1}{\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \left[\mathrm{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1]\right) + \mathrm{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)\right] & \\ \qquad + \frac{2}{M} \frac{1}{1+\mathrm{erf}\ \frac{\sqrt{E_0}}{\sqrt{2}\sigma}} \left[2\mathrm{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1]\right) + \mathrm{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x+1]\right) + \mathrm{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}\right)\right] & \text{for } 0 \le x < 1 \\ 1 & \text{for } x \ge 1 \\ 0 & \text{otherwise} . \end{cases} \tag{66}$$

Appendix E: Cumulative Probability Density Function of β

For the derivation of the CDF of β, we follow a similar methodology as in the last section. However, since the final obtain expression is quite lengthy, we do not present the overall CDF of β, leaving readers the trivial task of assembling this expression from the following conditional CDF expressions.

Recall that we have the following expression for the PDF of β

$$f_\beta(x) = \frac{1}{M} \sum_{m=0} M - 1 \frac{1}{1 - \Pr(r = m|s = m)} \sum_{\ell=0, \ell \neq m}^{M-1} \Pr(r = \ell|m) \, f_\beta(x|r = \ell, s = m) \ . \tag{67}$$

Therefore we write the required CDF as

$$F_\beta(x) = \frac{1}{M} \sum_{m=0} M - 1 \frac{1}{1 - \Pr(r = m|s = m)} \sum_{\ell=0, \ell \neq m}^{M-1} \Pr(r = \ell|m) \, F_\beta(x|r = \ell, s = m) \ , \tag{68}$$

where the conditional CDFs are defined as

$$F_\beta(x|r = \ell, s = m) = \int_{-\infty}^{x} f_\beta(y|r = \ell, s = m) \, dy = \int_{0}^{x} f_\beta(y|r = \ell, s = m) \, dy \ . \tag{69}$$

We now find each of the required conditional cdf's, by finding integrals of the conditional pdf's of the unclipped reliability, β', in the range $0 \leq x < 1$. These pdf's are given in the previous section. For $x \geq 1$ it is readily seen from the pdf's that $F_\beta(x)\Big|_{x \geq 1} = 1$.

For $r = \ell = 0$, we find the conditional CDF of β to be

$$
\begin{aligned}
&F_\beta(x|r=0,s=m) \\
&= \int_0^x f_\beta(y|r=0,s=m)\,dy \\
&= \begin{cases}
\frac{1}{\Pr(r=0|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \int_0^x \Big[\exp\left(-\frac{E_0}{2\sigma^2}[1-y-2m]^2\right) + \exp\left(-\frac{E_0}{2\sigma^2}[1-y+2m]^2\right) & \\
\qquad + \exp\left(-\frac{E_0}{2\sigma^2}[1+y+2m]^2\right)\Big]\,dy & \text{for } 0 \le x < 1 \\
1 & \text{for } x \ge 1 \\
0 & \text{otherwise,}
\end{cases} \\
&= \begin{cases}
\frac{1}{\Pr(r=0|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \frac{\sqrt{\pi}}{2\sqrt{\frac{E_0}{2\sigma^2}}} \Big[\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1+2m]\right) - \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0-1+2m]\right) + \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1-2m]\right) & \\
-\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0-1-2m]\right) + \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x+1+2m]\right) - \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0+1+2m]\right)\Big] & \text{for } 0 \le x \le 1 \\
1 & \text{for } x \ge 1 \\
0 & \text{otherwise,}
\end{cases} \\
&= \begin{cases}
\frac{1}{2\Pr(r=0|s=m)} \Big[\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1-2m]\right) + \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1+2m]\right) & \\
+\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x+1+2m]\right) + \operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[1-2m]\right)\Big] & \text{for } 0 \le x < 1 \\
1 & \text{for } x \ge 1 \\
0 & \text{otherwise;}
\end{cases}
\end{aligned}
$$

where we have made use of the expressions in (34) in making the above simplifications.

For the case where $r = \ell = M-1$, we find the conditional CDF of β to be

$$
\begin{aligned}
&F_\beta(x|r=M-1, s=m)\\
&= \int_0^x f_\beta(y|r=M-1,s=m)\,dy\\
&= \begin{cases}
\frac{1}{\Pr(r=M-1|s=m)}\frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma}\int_0^x \Big[\exp\left(-\frac{E_0}{2\sigma^2}[1-y-2(M-1-m)]^2\right)+\exp\left(-\frac{E_0}{2\sigma^2}[1-y+2(M-1-m)]^2\right) & \\
\qquad +\exp\left(-\frac{E_0}{2\sigma^2}[1+y+2(M-1-m)]^2\right)\Big]\,dy & \text{for } 0\le x<1\\
1 & \text{for } x\ge 1\\
0 & \text{otherwise,}
\end{cases}\\
&= \begin{cases}
\frac{1}{\Pr(r=M-1|s=m)}\frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma}\frac{\sqrt{\pi}}{2\sqrt{\frac{E_0}{2\sigma^2}}}\Bigg[\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1+2(M-1-m)]\right)-\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0-1+2(M-1-m)]\right) & \\
\qquad +\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1-2(M-1-m)]\right)-\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0-1-2(M-1-m)]\right) & \\
\qquad +\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x+1+2(M-1-m)]\right)-\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0+1+2(M-1-m)]\right)\Bigg] & \text{for } 0\le x<1\\
1 & \text{for } x\ge 1\\
0 & \text{otherwise,}
\end{cases}\\
&= \begin{cases}
\frac{1}{2\Pr(r=M-1|s=m)}\Bigg[\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1-2(M-1-m)]\right)+\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1+2(M-1-m)]\right) & \\
\qquad +\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x+1+2(M-1-m)]\right)+\operatorname{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[1-2(M-1-m)]\right)\Bigg] & \text{for } 0\le x<1\\
1 & \text{for } x\ge 1\\
0 & \text{otherwise;}
\end{cases}
\end{aligned}
\tag{70}
$$

where once again we use the expressions in (34) for the above simplification.

Finally, we consider the case when the received hard decision point is $p^r = p^\ell$, for some $\ell \in$

$\{1, 2, \ldots, M-2\}$. In this case, we find the conditional CDF of β to be

$$
\begin{aligned}
&F_\beta(x|r=\ell, s=m) \\
&= \int_0^x f_\beta(y|r=\ell, s=m)\,dy \\
&= \begin{cases}
\frac{1}{\Pr(r=\ell|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \int_0^x \Big[\exp\left(-\frac{E_0}{2\sigma^2}[1-y-2(\ell-m)]^2\right) + \exp\left(-\frac{E_0}{2\sigma^2}[1-y+2(\ell-m)]^2\right) + \\
\qquad \exp\left(-\frac{E_0}{2\sigma^2}[1+y+2(\ell-m)]^2\right)\Big]\,dy \qquad \text{for } 0 \le x < 1 \\
1 \qquad \text{for } x \ge 1 \\
0 \qquad \text{otherwise,}
\end{cases} \\
&= \begin{cases}
\frac{1}{\Pr(r=\ell|s=m)} \frac{\sqrt{E_0}}{\sqrt{2\pi}\sigma} \frac{\sqrt{\pi}}{2\sqrt{\frac{E_0}{2\sigma^2}}} \Bigg[\text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1+2(\ell-m)]\right) - \text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0-1+2(\ell-m)]\right) \\
\qquad +\text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1-2(\ell-m)]\right) - \text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0-1-2(\ell-m)]\right) \\
\qquad +\text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x+1+2(\ell-m)]\right) - \text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[0+1+2(\ell-m)]\right)\Bigg] \qquad \text{for } 0 \le x < 1 \\
1 \qquad \text{for } x \ge 1 \\
0 \qquad \text{otherwise,}
\end{cases} \\
&= \begin{cases}
\frac{1}{2\Pr(r=\ell|s=m)} \Bigg[\text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1-2(\ell-m)]\right) + \text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x-1+2(\ell-m)]\right) \\
\qquad +\text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[x+1+2(\ell-m)]\right) + \text{erf}\left(\frac{\sqrt{E_0}}{\sqrt{2}\sigma}[1-2(\ell-m)]\right)\Bigg] \qquad \text{for } 0 \le x < 1 \\
1 \qquad \text{for } x \ge 1 \\
0 \qquad \text{otherwise;}
\end{cases}
\end{aligned} \tag{71}
$$

We can then obtain the final expression for the CDF of β by substituting the above three conditional CDF functions for β into (68). Since this task is trivial, yet yields a lengthy result, we omit displaying the final CDF of β.

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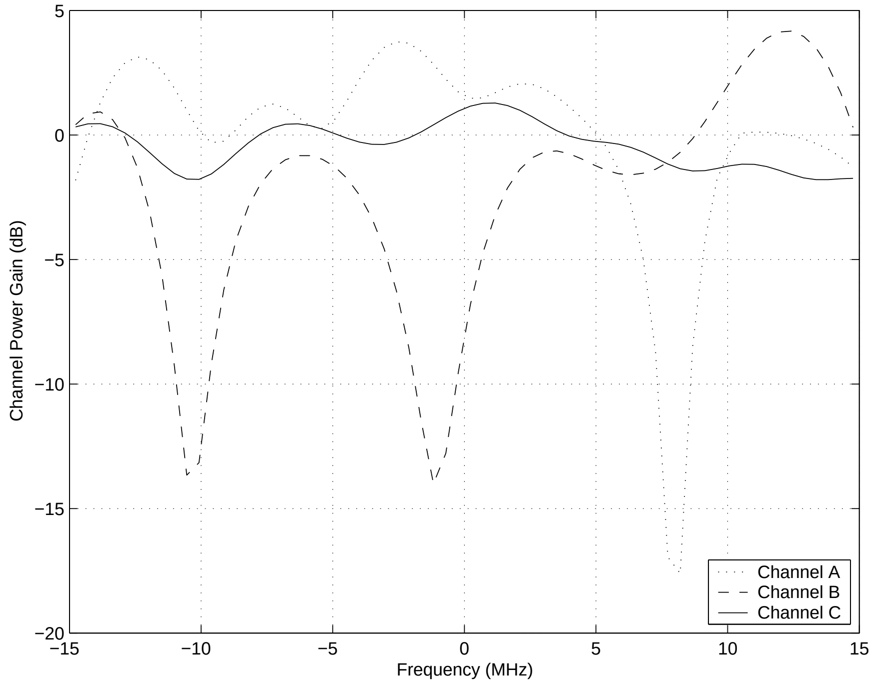

Figure 1: Channel Gains for Channels A, B and C

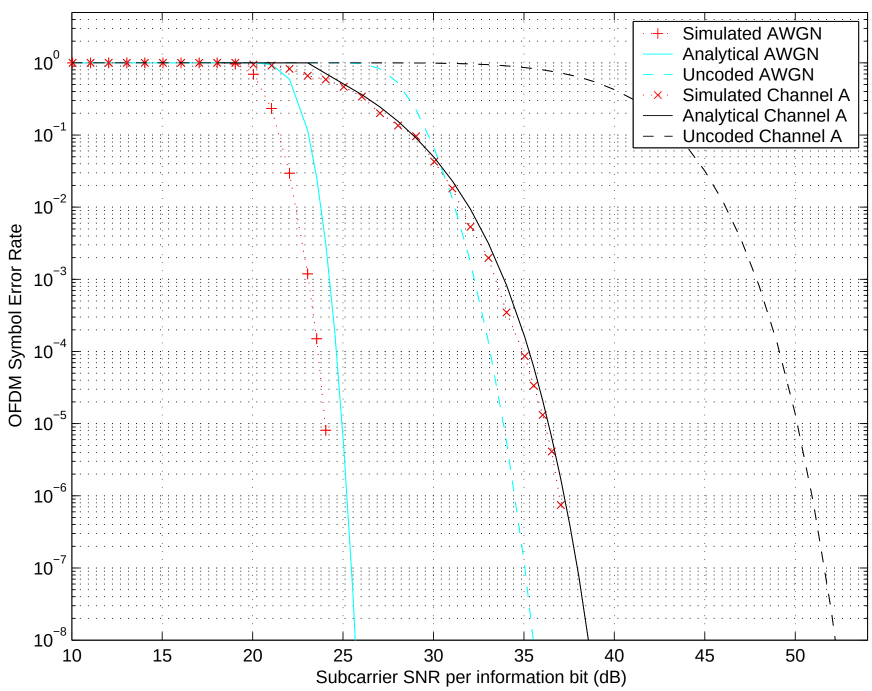

Figure 2: BW_{128} Lattice Encoded, GMD Decoded, 256-QAM , 64 Subcarrier OFDM System: Error Rates for AWGN and Channel A

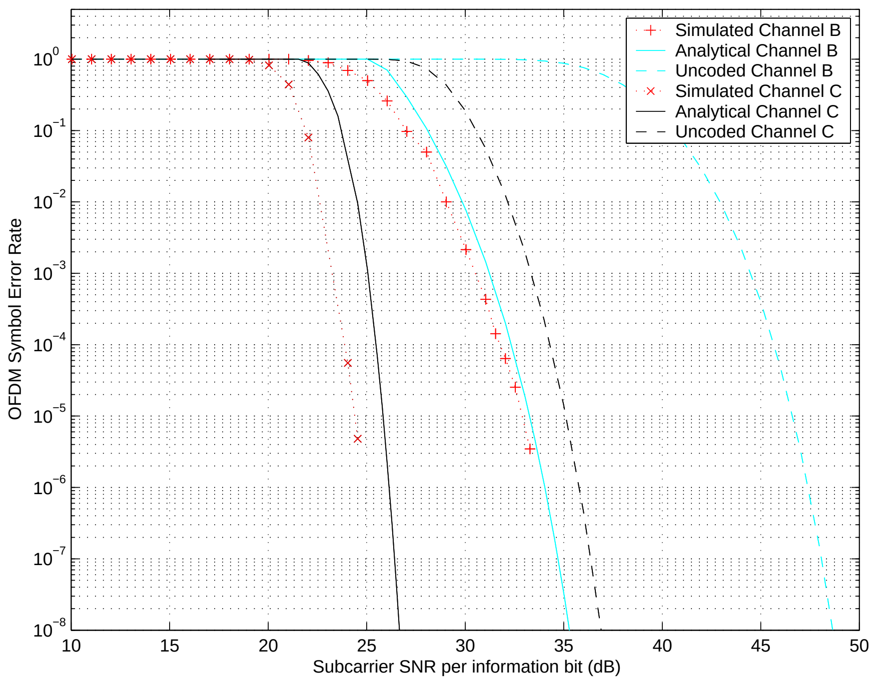

Figure 3: BW_{128} Lattice Encoded, GMD Decoded, 256-QAM , 64 Subcarrier OFDM System: Error Rates for Channel B and Channel C